

ON BLASCHKE PRODUCTS WITH DERIVATIVES IN BERGMAN SPACES WITH NORMAL WEIGHTS

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ABSTRACT. We generalize a well-known sufficient condition for interpolating sequences for the Hilbert Bergman spaces to other Bergman spaces with normal weights (as defined by Shields and Williams) and obtain new results regarding the membership of the derivative of a Blaschke product or a general inner function in such spaces. We also apply duality techniques to obtain further results of this type and obtain new results about interpolating Blaschke products.

INTRODUCTION

The conditions under which the derivative of a Blaschke product belongs to certain spaces of analytic functions in the disk is a subject with long history. Membership of the derivative in the Hardy spaces H^p was investigated in [26], [3], [1], and [7] and a study of when it belongs to B^p , the Banach envelope of H^p with $0 < p < 1$, can be found in [3, 4]. For an overview of such results until the early 1980s, we refer the reader to [8].

Membership of the derivative of a Blaschke product in weighted Bergman spaces $L_a^{p,\alpha}$ was studied in [2] and [20] and more recently in [21], [15, 17, 16], [13, 14], and [11]. Recall that for $\alpha > -1$ the *standard weighted Bergman space* $L_a^{p,\alpha}$ (often also denoted by A_α^p) is defined as the set of all analytic functions f in the unit disk \mathbb{D} for which

$$\|f\|_{L_a^{p,\alpha}}^p = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty.$$

When $\alpha = 0$, we write simply L_a^p and refer to it as the *Bergman space*. A direct application of the Schwarz-Pick lemma shows that the derivative of any Blaschke product B belongs to $\cap_{0 < p < 1} L_a^p$. Rudin [27] proved that there are Blaschke products whose derivative does not belong to L_a^1 and Piranian [25] gave an explicit example. Ahern [2] found a necessary and

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sufficient condition for the membership of B' in the standard weighted Bergman space $L_a^{p,\alpha}$ expressed in terms of $|B|$.

Since Ahern's condition is non-trivial to check, it is desirable to have other conditions, either necessary or sufficient, for the membership of B' in L_a^p or in more general weighted Bergman space $L_a^p(w)$, where by a *weight* w we mean any strictly positive measurable function w in \mathbb{D} which replaces the function $(\alpha + 1)(1 - |z|^2)^\alpha$ in the definition. Such criteria are often given in terms of the moduli of the zeros of B or in terms of their location: belonging to some Stolz angle, some separation (discreteness) condition, etc. It is precisely this setting in which a study was carried out in [20], [13, 14], [15, 17, 16], or [11].

Recall that a sequence $\{a_n\}$ in \mathbb{D} is said to be *separated* (or *uniformly discrete*) with *constant of separation* δ if

$$(1) \quad \inf_{n \in \mathbb{N}} \prod_{k \neq n} \left| \frac{a_k - a_n}{1 - \bar{a}_k a_n} \right| = \delta > 0.$$

By a well-known theorem of Carleson, this is equivalent to $(a_n)_{n=1}^\infty$ being an interpolating sequence for the space H^∞ of bounded analytic functions. An important class of separated sequences is that of exponential sequences; it is known that, unlike in Rudin's example, the derivative of every Blaschke product whose zeros form an exponential sequence belongs to L_a^1 .

The main purpose of this article is to extend several earlier results to the context of weighted Bergman spaces with normal weights as defined by Shields and Williams [31]. For a non-negative radial weight on the unit disk \mathbb{D} , its values in \mathbb{D} are completely determined by the values in the interval $[0, 1)$. A positive function $w(r)$ in this interval is normal if, roughly speaking, it grows at the rate controlled by two positive powers of $1 - r$ (see [31] and [5], for example). As a "rule of thumb", a space with such weights behaves to a certain extent like the unweighted one.

In Section 1 of the paper we review the necessary background and prove two results. One is a basic proposition of Shields-Williams type which explains the asymptotic behavior of certain weighted integrals and which is needed in the rest. The other is a sufficient condition for interpolation in weighted Bergman spaces. Our main tool in this section is the use of the so-called Békollé condition [6].

In Section 2 we consider some conditions which are either necessary or sufficient for membership of the derivative of a Blaschke product in Bergman spaces with normal weights, thus generalizing several results obtained previously. We also study the integrability of the derivative of a general inner function in the disk in terms of the properties of its generalized counting functions. Our results allow us to deduce as a corollary one of the main results from [11]. Also, several propositions from [15] follow for free as well. It follows from our main theorems that there are

interpolating Blaschke products which are in no L_a^p , $p > 1$. An explicit example was given in [13, Theorem 3.8]. We remark that Peláez [24], inspired by Piranian's construction [25], has recently constructed such a product which is not in L_a^1 . It is not clear whether our method could allow us to produce such examples, so there are some limitations to the techniques used here.

In Section 3 we use Luecking's duality theorem for weighted Bergman spaces to obtain some further estimates on the integral of $|B'|$. As a main result, our Theorem 4 provides the weighted Bergman space analogue of a theorem due to Cohn [7] which relates the membership of $|B'|$ in certain Hardy spaces with the orthogonal complement of the invariant subspace of H^2 generated by B .

Section 4 includes some closing remarks. In it, we mention some possible formulations of our results in terms of derivatives of higher order.

1. NORMAL WEIGHTS

We will typically consider measurable functions $w : \mathbb{D} \rightarrow [0, +\infty)$. Such a function is said to be *radial* if $w(z) = w(|z|)$ for all z in \mathbb{D} ; obviously, its values are then completely determined by the values on the interval $[0, 1)$. Since such functions are usually important examples of weights for Bergman spaces, further special growth or integrability conditions are often imposed on them, especially in relation to certain integral operators; see [5], for example. In this paper we will also consider one special and important class of radial weights: the so-called normal ones. Following Shields and Williams [31], we will say that the function $w : [0, 1) \rightarrow [0, \infty)$ is *normal* if there exist real numbers a and b and $r_0 \in (0, 1)$ for which the two following conditions are fulfilled:

$$\frac{w(r)}{(1-r)^a} \nearrow \infty \quad \text{for } r > r_0$$

and

$$\frac{w(r)}{(1-r)^b} \searrow 0 \quad \text{for } r > r_0.$$

One of the basic properties of radial normal functions is the following fact.

LEMMA 1. *For each $t \in (0, 1)$ there exist constants c and C depending only on t such that*

$$c w(\zeta) \leq w(z) \leq C w(\zeta)$$

whenever $|z| > r_0$ and $|z - \zeta| < t(1 - |z|)$.

Proof. The proof is simple, so to illustrate the method it suffices to prove only one of the inequalities, say in the case $|\zeta| \geq |z|$. Under this assumption, the obvious inequalities

$$\frac{w(|z| + t(1 - |z|))}{(1 - t)^b(1 - |z|)^b} = \frac{w(|z| + t(1 - |z|))}{(1 - |z| - t(1 - |z|))^b} \leq \frac{w(|z|)}{(1 - |z|)^b} = \frac{w(z)}{(1 - |z|)^b}$$

yield

$$w(\zeta) = w(|\zeta|) \leq w(|z| + t(1 - |z|)) \leq (1 - t)^b w(z)$$

whenever $|z - \zeta| < t(1 - |z|)$. \square

Given a normal function w , denote by a_w and b_w the optimal indices

$$a_w = \inf\{a : \frac{w(r)}{(1 - r)^a} \nearrow \infty, r > r_a\}, \quad b_w = \sup\{b : \frac{w(r)}{(1 - r)^b} \searrow 0, r > r_b\}.$$

Note that $a_w \geq b_w$ and also that it is not required that the functions

$$\frac{w(r)}{(1 - r)^{a_w}}, \quad w(r)(1 - r)^{b_w}$$

be increasing and decreasing respectively.

We list below some estimates for normal functions. They should be compared with the usual integral estimates for the standard radial weights. For example, it is well known that

$$I_{\gamma, m}(\lambda) \asymp \int_{\mathbb{D}} \frac{(1 - |z|)^\gamma}{|1 - \bar{\lambda}z|^{m+1}} dA(z) = O((1 - |\lambda|)^{\gamma - m + 1})$$

as $|\lambda| \rightarrow 1$ (see [18, Theorem 1.7], for example). As is usual, throughout the paper the notation $u(\lambda) \asymp v(\lambda)$ will mean that the quotient of two positive functions u and v is bounded from above and below as $|\lambda| \rightarrow 1$.

The estimates below are essentially known to the experts and can be attributed to Shields and Williams. Certainly, some lemmas in their paper [31] have a similar flavor and use similar proofs.

PROPOSITION 1. *Let w be a normal function and let $m \in \mathbb{R}$. For $\lambda \in \mathbb{D}$, let*

$$I_{m, w}(\lambda) = \int_{\mathbb{D}} \frac{w(|z|)}{|1 - \bar{\lambda}z|^{m+2}} dA(z),$$

$$J_{m, w}(\lambda) = \int_{\mathbb{D}} \log \left| \frac{1 - \bar{\lambda}z}{\lambda - z} \right| (1 - |z|)^{-m-2} w(|z|) dA(z).$$

(i) *If $m > -1$, $a_w < m$, and $b_w > -1$ then*

$$I_{m, w}(\lambda) \asymp w(|\lambda|)(1 - |\lambda|)^{-m}.$$

(ii) *If $m \in \mathbb{R}$, $a_w < m + 1$, and $b_w > m$ then*

$$J_{m, w}(\lambda) \asymp w(|\lambda|)(1 - |\lambda|)^{-m}.$$

Assume also that:

(*) There exists α such that $0 < \alpha < 1$ and the function

$$\frac{w(r)}{(1-r)^{a_w}} \log^\alpha \frac{1}{1-r}$$

is increasing for $r > r_0$.

Then:

(iii) If $m > -1$, $a_w = m$ and $b_w > -1$

$$I_{m,w}(\lambda) = O \left(w(|\lambda|)(1-|\lambda|)^{-m} \log \frac{1}{1-|\lambda|} \right) \quad \text{when } |\lambda| \rightarrow 1.$$

(iv) If $m \in \mathbb{R}$, $a_w = m + 1$, and $b_w > m$ then

$$J_{m,w}(\lambda) = O \left(w(|\lambda|)(1-|\lambda|)^{-m} \log \frac{1}{1-|\lambda|} \right) \quad \text{as } |\lambda| \rightarrow 1.$$

Proof. We only prove parts (i) and (ii) in order to illustrate the basic technique. Let a, b be arbitrary real numbers such that $m > a > a_w$ and $-1 < b < b_w$.

(i) By applying the usual splitting argument for the normal weights and taking into account that the function $(1-r)^{-a}w(r)$ is increasing on $(r_0, 1)$ and $(1-r)^{-b}w(r)$ is decreasing on $(r_0, 1)$, we obtain for $r_0 < |\lambda| < 1$

$$\begin{aligned} I_{m,w}(\lambda) &= \int_{\mathbb{D}} \frac{w(z)}{|1 - \bar{\lambda}z|^{m+2}} dA(z) = \int_0^1 r w(r) \left(\int_0^{2\pi} \frac{d\theta}{|1 - \bar{\lambda}z|^{m+2}} \right) \frac{dr}{\pi} \\ &\leq C \int_{r_0}^1 \frac{r w(r)}{(1-|\lambda|r)^{m+1}} dr \\ &\leq C \left(\int_{r_0}^{|\lambda|} \frac{w(r)}{(1-|\lambda|r)^{m+1}} dr + \int_{|\lambda|}^1 \frac{w(r)}{(1-|\lambda|r)^{m+1}} dr \right) \\ &\leq C(1-|\lambda|)^{-a} w(|\lambda|) \int_{r_0}^{|\lambda|} \frac{(1-r)^a}{(1-|\lambda|r)^{m+1}} dr \\ &\quad + C(1-|\lambda|)^{-b} w(|\lambda|) \int_{|\lambda|}^1 \frac{(1-r)^b}{(1-|\lambda|r)^{m+1}} dr \\ &\leq C(1-|\lambda|)^{-a} w(|\lambda|) \int_0^{|\lambda|} (1-r)^{a-m-1} \\ &\quad + (1-|\lambda|)^{-b-m-1} w(|\lambda|) \int_{|\lambda|}^1 (1-r)^b dr \\ &\leq C w(|\lambda|)(1-|\lambda|)^{-m}. \end{aligned}$$

The constant C above may differ from one appearance to another. Similarly, since $-1 < b < b_w \leq a_w < m$,

$$\begin{aligned} I_{m,w}(\lambda) &\geq C \int_{r_0}^{|\lambda|} \frac{w(r)}{(1-|\lambda|r)^{m+1}} dr \geq C(1-|\lambda|)^{-b} w(|\lambda|) \int_{r_0}^{|\lambda|} \frac{(1-r)^b}{(1-|\lambda|r)^{m+1}} \\ &\geq C(1-|\lambda|)^{-m} w(|\lambda|) \int_1^{\frac{1-r_0}{1-|\lambda|}} \frac{x^b}{(1+x)^{m+1}} dx \geq C(1-|\lambda|)^{-m} w(|\lambda|). \end{aligned}$$

(ii) By Jensen's formula we have

$$J_{m,w}(\lambda) = \log \frac{1}{|\lambda|} \int_0^{|\lambda|} w(r)(1-r)^{-m-2} r dr + \int_{|\lambda|}^1 \log \frac{1}{r} w(r)(1-r)^{-m-2} r dr,$$

and the estimates for $J_{m,w}(\lambda)$ follow with the same argument as above. \square

If the normal weight w is integrable, for example, if $b_w > -1$, we can consider the weighted Bergman space $L_a^p(w)$, $p > 0$, which consists of analytic functions f in \mathbb{D} with

$$\|f\|_{p,w}^p = \int_{\mathbb{D}} |f|^p w dA < \infty.$$

Using Lemma 1, it follows easily that if $f \in L_a^p(w)$ then

$$|f(z)| \leq C (w(|z|)(1-|z|)^2)^{-1/p} \|f\|_{p,w},$$

for some constant $C > 0$ and all $z \in \mathbb{D}$. By Proposition 1 (i) we have that

$$\sup_{\|f\|_{p,w} \leq 1} |f(z)| \asymp (w(|z|)(1-|z|)^2)^{-1/p},$$

if $b_w > -1$. We can then consider interpolating sequences for these spaces. A sequence $(z_n)_{n=1}^\infty$ in \mathbb{D} is called an *interpolating sequence* for $L_a^p(w)$ if for every $(a_n) \in \ell^p$ there exists $f \in L_a^p(w)$ such that

$$f(z_n) (w(|z_n|)(1-|z_n|)^2)^{1/p} = a_n,$$

for all n . If $p = 2$, it is well known that interpolating sequences for Hardy spaces are interpolating for $L_{a,w}^2$ as well. (For example, this follows from [29, p. 23] by recalling the fact which is a “folk knowledge”: the pointwise multipliers of $L_{a,w}^2$ into itself are precisely the bounded analytic functions.) This interpolation result can be extended for the other values of $p > 1$.

THEOREM 1. *If $b_w > -1$, then every interpolating sequence for H^∞ is interpolating for $L_a^p(w)$, $p > 1$.*

Proof. Our proof uses Békollé's theorem [6] and Luecking's duality theorem [22]. To be more explicit, if $\frac{1}{p} + \frac{1}{q} = 1$, let $\gamma > \frac{aw}{p} - \frac{1}{q}$ and use a similar argument to the above to obtain that

$$\left(\int_r^1 w(t) dt \right) \left(\int_r^1 (1-t)^{\gamma q} w^{-q/p}(t) dt \right)^{p/q} \leq C(1-r)^{p\gamma+p},$$

which implies that $w(|z|)(1-|z|)^{-\gamma}$ belongs to the Békollé class $B_p(\gamma)$, or equivalently, $v(z) = w^{-q/p}(z)(1-|z|)^{(q-1)\gamma}$ belongs to $B_q(\gamma)$. Then by a result in [6] it follows that the sublinear operator P_γ defined by

$$(2) \quad P_\gamma f(z) = \int_{\mathbb{D}} \frac{(1-|\zeta|)^\gamma}{|1-\bar{\zeta}z|^{\gamma+2}} |f(\zeta)| dA(\zeta),$$

is bounded from $L^q((1-|z|)^{q\gamma} w^{-q/p} dA)$ into itself. The duality theorem in [22] then identifies the dual of $L_a^p(w)$ with $L_a^q(v)$ via the standard weighted Bergman pairing with index γ . In other words, the norm in $L_a^p(w)$ is equivalent to

$$(3) \quad \sup \left\{ \left| \int_{\mathbb{D}} f(z) \overline{g(z)} (1-|z|^2)^\gamma dA \right| : g \in L_a^q(w), \|g\|_{q,v} \leq 1 \right\}.$$

Now let $(z_n)_{n=1}^\infty$ be an interpolating sequence for H^∞ , let B be the Blaschke product with zeros z_1, \dots, z_n, \dots and let $a = (a_n) \in \ell^p$. We claim that the solution f_a of the interpolation problem defined above is given by

$$f_a(z) = \sum_n a_n (w(|z_n|)(1-|z_n|)^2)^{-1/p} \frac{(1-|z_n|^2)^{\gamma+1} B(z)}{(1-\bar{z}_n z)^{\gamma+1} (z-z_n) B'(z_n)}.$$

In order to verify this, it suffices to show that

$$\|f_a\|_{p,w} \leq C \|a\|_{\ell^p},$$

for some constant C and all sequences $a \in \ell^p$ with finitely many nonzero terms. Denote by B_n the Blaschke product with zeros $\{z_k : k \neq n\}$ taking multiplicities into account, and recall that the values $|B_n(z_n)|$ are bounded away from zero by (1). By (3) we have

$$\begin{aligned} & \|f_a\|_{p,w} \leq \\ & \sup_{\|g\|_{q,v} \leq 1} \sum_n \frac{|a_n|}{|B_n(z_n)|} w^{-1/p}(|z_n|)(1-|z_n|)^{\gamma+2/q} \int_{\mathbb{D}} \frac{|g(z)| |B_n(z)| (1-|z|)^\gamma}{|1-\bar{z}_n z|^{\gamma+2}} dA(z) \\ & \leq C \sup_{\|g\|_{q,v} \leq 1} \sum_n |a_n| w^{-1/p}(|z_n|)(1-|z_n|)^{\gamma+2/q} P_\gamma g(z_n) \\ & \leq C \|a\|_{\ell^p} \sup_{\|g\|_{q,v} \leq 1} \left(\sum_n w^{-q/p}(|z_n|)(1-|z_n|)^{q\gamma+2} P_\gamma^q g(z_n) \right)^{1/q}. \end{aligned}$$

Since $(z_n)_{n=1}^\infty$ is separated, there is a fixed $R \in (0, 1)$ such that the disks

$$\Delta_n = \{\zeta \in \mathbb{D} : |\zeta - z_n| < R(1-|z_n|)\},$$

are pairwise disjoint. Moreover, it is easy to show that $P_\gamma g$ is almost constant on these disks, *i.e.*, there exists $c_0 > 0$ independent of g such that

$$c_0^{-1} P_\gamma g(z) \leq P_\gamma g(\zeta) \leq c_0 P_\gamma g(z), \quad z, \zeta \in \Delta_n.$$

This, together with Lemma 1, leads to

$$\begin{aligned} & \sup_{\|g\|_{q,v} \leq 1} \sum_n w^{-q/p}(|z_n|)(1 - |z_n|)^{q\gamma+2} P_\gamma^q g(z_n) \\ & \leq C \sup_{\|g\|_{q,v} \leq 1} \sum_n w^{-q/p}(|z_n|)(1 - |z_n|)^{q\gamma} \int_{\Delta_n} P_\gamma^q g(z) dA(z) \\ & \leq C \sup_{\|g\|_{q,v} \leq 1} \sum_n \int_{\Delta_n} P_\gamma^q g(z) w^{-q/p}(z)(1 - |z|)^{q\gamma} dA(z) \\ & \leq \sup_{\|g\|_{q,v} \leq 1} \int_{\cup_n \Delta_n} P_\gamma^q g(z) w^{-q/p}(z)(1 - |z|)^{q\gamma} dA(z). \end{aligned}$$

and the result follows by (2). \square

2. MAIN RESULTS

In what follows, for a nonconstant analytic function f in \mathbb{D} and a function $u : \mathbb{D} \rightarrow \mathbb{R}_+$ we will consider the sum

$$\sum_{f(z)=\zeta} u(z),$$

the multiplicities being taken into account. In particular, such sums will be useful to denote the summation over all the zeros of a Blaschke product. We remind the reader that most results obtained in the literature on the integrability of B' are obtained under certain special assumptions on the distribution or moduli of the zeros of B .

Our first statement in this section generalizes or adds new information to several known results. Besides complementing the well-known statement (attributed to Rudin) that

$$\|B'\|_1 \leq c \sum_{B(z)=0} (1 - |z|) \log \frac{1}{1 - |z|},$$

it also includes as special cases several propositions from the recent papers such as [15] and [11].

THEOREM 2. *Let w be a normal function and let $1/2 < p < \infty$.*

- (i) *Assume that $a_w < 2p - 2$, $b_w > -1$, if $1/2 < p \leq 1$, and that $a_w < p - 1$, $b_w > p - 2$, if $p > 1$. Then there exists a positive constant $c_{p,w}$ such that for every Blaschke product B we have*

$$(4) \quad \|B'\|_{p,w}^p \leq c_{p,w} \sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|).$$

If $1/2 < p \leq 1$ and $a_w = 2p - 2$, $b_w > -1$, or if $p > 1$, $a_w = p - 1$, $b_w > p - 2$, and w satisfies the condition (*) in Proposition 1, then there exists a positive constant $c_{p,w}$ such that for every Blaschke product B we have

$$(5) \quad \|B'\|_{p,w}^p \leq c_{p,w} \sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|) \log \frac{1}{1 - |z|}.$$

(ii) Suppose that the zero set of the Blaschke product B is separated with separation constant $\delta > 0$. Suppose that either $a_w < 2p - 2$, $b_w > -1$, when $1/2 < p \leq 1$, or $a_w < p - 2$, $b_w > -1$, when $p > 1$. Then, in both cases, there exists a positive constant $c_{p,w,\delta}$ such that

$$(6) \quad \sum_{B(z)=0}^{\infty} (1 - |z|)^{2-p} w(|z|) \leq c_{p,w,\delta} \|B'\|_{p,w}^p.$$

If either $1/2 < p \leq 1$ and $a_w = 2p - 2$, $b_w > -1$, or $p > 1$, $a_w = p - 1$, $b_w > p - 2$, and w also satisfies the condition (*) in Proposition 1, then there exists a positive constant $c_{p,w,\delta}$ such that for every Blaschke product B we have

$$(7) \quad \sum_{B(z)=0}^{\infty} (1 - |z|)^{2-p} w(|z|) \leq c_{p,w,\delta} \int_{\mathbb{D}} |B'(z)|^p w(|z|) \log \frac{1}{1 - |z|} dA(z).$$

Proof. To prove (i) we assume first that $1/2 < p \leq 1$. Start off with the inequality

$$|B'(\zeta)| \leq \sum_{B(z)=0} \frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2},$$

which is quite straightforward to deduce (see [11], for example). For $p \leq 1$ this implies

$$\int_{\mathbb{D}} |B'|^p w dA \leq \sum_{B(z)=0} (1 - |z|^2)^p \int_{\mathbb{D}} \frac{w(\zeta)}{|1 - \bar{z}\zeta|^{2p}} dA(z).$$

Thus for $1/2 < p \leq 1$, part (i) follows by a direct application of Theorem 1 (i) and (iii), with $m = 2p - 2$.

In the case $p > 1$ we can apply the Schwarz-Pick lemma together with the elementary inequality $(1 - x^2)^p \leq -2 \log x$, $x \in [0, 1]$, to obtain

$$\begin{aligned}
\int_{\mathbb{D}} |B'|^p w \, dA &\leq \int_{\mathbb{D}} \frac{(1 - |B(\zeta)|^2)^p}{(1 - |\zeta|^2)^p} w(|\zeta|) \, dA(\zeta) \\
&\leq -2 \int_{\mathbb{D}} \log |B(\zeta)| w(|\zeta|) (1 - |\zeta|^2)^{-p} \, dA(\zeta) \\
&= 2 \sum_{B(z)=0} \int_{\mathbb{D}} \log \left| \frac{1 - \bar{z}\zeta}{z - \zeta} \right| (1 - |\zeta|)^{-p} w(|\zeta|) \, dA(\zeta) \\
&= 2 \sum_{B(z)=0} J_{p-2,w}(z),
\end{aligned}$$

and the result follows by Theorem 1 (ii) and (iv).

(ii) Assume that the zero set of B is separated with the constant of separation $\delta > 0$. Then there is a fixed $R \in (0, 1)$ depending only on δ such that the disks

$$\Delta_z = \{\zeta \in \mathbb{D} : |\zeta - z| < R(1 - |z|)\}, \quad B(z) = 0,$$

are pairwise disjoint. Moreover,

$$\rho(R) = \sup\{|B(\zeta)| : \zeta \in \cup_{B(z)=0} \Delta_z\} < 1.$$

See, for example, [15, pp. 681–682] or [23].

By Lemma 1, the obvious estimate for the area of the disk Δ_z , and an application of the standard Green's formula (see, e.g., [10, p. 16] for a formulation in the smooth case), and a routine Laplacian computations, for every $\eta \geq 0$ we have

$$\begin{aligned}
(1 - \rho(R)^2)^\eta \sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|) \\
&\leq C \sum_{B(z)=0} \int_{\Delta_z} (1 - |B(\zeta)|^2)^\eta w(|\zeta|) (1 - |\zeta|)^{-p} \, dA(\zeta) \\
&\leq C \int_{\mathbb{D}} (1 - |B(\zeta)|^2)^\eta w(|\zeta|) (1 - |\zeta|)^{-p} \, dA(\zeta) \\
&= -\frac{C}{2\pi} \int_{\mathbb{D}} \Delta(1 - |B(\xi)|^2)^\eta \int_{\mathbb{D}} \log \left| \frac{1 - \bar{\xi}\zeta}{\zeta - \xi} \right| w(|\zeta|) (1 - |\zeta|)^{-p} \, dA(\zeta) \, dA(\xi) \\
&\leq \frac{2\eta^2 C}{\pi} \int_{\mathbb{D}} (1 - |B(\xi)|^2)^{\eta-2} |B'(\xi)|^2 J_{p-2,w}(\xi) \, dA(\xi) \\
&\leq \frac{2\eta^2 CK}{\pi} \int_{\mathbb{D}} (1 - |B(\xi)|^2)^{\eta-2} |B'(\xi)|^2 (1 - |\xi|)^{2-p} w(|\xi|) \, dA(\xi),
\end{aligned}$$

where the last inequality follows from Proposition 1 (ii). Now if $p \leq 2$, we choose $\eta = 2$ and apply again the Schwarz-Pick lemma to obtain

$$(1 - \rho(R)^2)^\eta \sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|) \leq \frac{8CK}{\pi} \int_{\mathbb{D}} |B'(\xi)|^p w(|\xi|) dA(\xi).$$

In the case when $p > 2$, we choose $\eta = 2 + \frac{p-2}{p}$ and apply Hölder's inequality (in addition to an argument used earlier in the proof of part (i) and Proposition 1 (ii)) to obtain from above

$$\begin{aligned} & (1 - \rho(R)^2)^\eta \sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|) \\ & \leq C_p \|B'\|_{p,w}^{\frac{2}{p}} \left(\int_{\mathbb{D}} (1 - |B(\xi)|^2) (1 - |\xi|)^{-p} w(|\xi|) dA(\xi) \right)^{\frac{p-2}{p}} \\ & \leq C'_p \|B'\|_{p,w}^{\frac{2}{p}} \left(-2 \int_{\mathbb{D}} \log |B(\xi)| (1 - |\xi|)^{-p} w(|\xi|) dA(\xi) \right)^{\frac{p-2}{p}} \\ & \leq C''_p \|B'\|_{p,w}^{\frac{2}{p}} \left(\sum_{B(z)=0} J_{p-2,w}(z) \right)^{\frac{p-2}{p}} \\ & \leq C'''_p \|B'\|_{p,w}^{\frac{2}{p}} \left(\sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|) \right)^{\frac{p-2}{p}}, \end{aligned}$$

and the first estimate follows. The proof of the second inequality in (ii) is almost identical and will be omitted. \square

Let us observe right away that one of the main results of Fricain and Mashregi [11] is a direct consequence of Theorem 2 (i). In fact, our result together with the obvious estimates

$$\begin{aligned} \frac{1}{\varepsilon} \int_{|z| < 1/2} |B'(z)|^p w(|z|) (1 - |z|)^\varepsilon dA(z) & \leq \sup_{1/2 < r < 1} w(r) (1 - r) \int_0^{2\pi} |B'(re^{it})|^p dt \\ & \leq 2 \int_{|z| < 1/2} |B'(z)|^p w(|z|) dA(z), \end{aligned}$$

where $\varepsilon > 0$, lead to the following result.

COROLLARY 1. *If w and p satisfy the conditions in Theorem 2 (i) then there exists a constant $c_{p,w} > 0$ such that*

$$\sup_{1/2 < r < 1} w(r) (1 - r) \int_0^{2\pi} |B'(re^{it})|^p dt \leq \sum_{B(z)=0} (1 - |z|)^{2-p} w(|z|),$$

when $a_w < 2p - 2$, $1/2 < p \leq 1$, or $a_w < p - 1$, $p > 1$, and

$$\sup_{1/2 < r < 1} w(r)(1-r) \int_0^{2\pi} |B'(re^{it})|^p dt \leq \sum_{B(z)=0} (1-|z|)^{2-p} w(|z|) \log \frac{1}{1-|z|},$$

when $a_w = 2p - 2$, $1/2 < p \leq 1$, or $a_w = p - 1$, $p > 1$.

If w and p satisfy the conditions in Theorem 2 (ii) and the zero set of the Blaschke product B is separated with separation constant $\delta > 0$, then for every $\varepsilon > 0$ there exists a positive constant $c_{p,w,\delta,\varepsilon}$ such that

$$\sum_{n=1}^{\infty} (1-|z|)^{2-p+\varepsilon} w(|z|) \leq c_{p,w,\delta,\varepsilon} \sup_{1/2 < r < 1} w(r)(1-r) \int_0^{2\pi} |B'(re^{it})| dt.$$

The lower estimate given in the second part of Theorem 2 cannot be expected to hold for general Blaschke products. For example, if $p = 1$ and B is a finite Blaschke product then it can be seen that $\|(B^n)'\|_{1,w} = o(n)$, as $n \rightarrow \infty$, while

$$\sum_{B^n(z)=0} (1-|z|)w(|z|) = n \sum_{B(z)=0} (1-|z|)w(|z|),$$

for any normal weight w . From this point of view, the limit cases $a_w = 2p - 2$, $1/2 < p \leq 1$, and $a_w = p - 1$, $p > 1$, raise even more delicate problems.

It turns out that an asymptotically sharp estimate of the integrals considered here can be obtained even for general inner functions, if we consider pre-images of several points in the disk. To this end we introduce, for a normal weight w , $p > 0$ and analytic functions f in \mathbb{D} , the *generalized counting functions*

$$(8) \quad N_{f,w,p}(\zeta) = \sum_{f(z)=\zeta} (1-|z|)^{2-p} w(|z|),$$

and

$$(9) \quad N_{f,w,p}^{\ell}(\zeta) = \sum_{f(z)=\zeta} (1-|z|)^{2-p} w(|z|) \log \frac{1}{1-|z|}.$$

This notation allows us to formulate a result on integrability of the derivative of a general inner function rather than just for a Blaschke product.

THEOREM 3. *Let w be a normal function and let $1/2 < p < \infty$. Then there exists a positive constant $c_{p,w}$ such that for every inner function θ and every $\varepsilon \in (0, 1/2)$ we have*

$$(10) \quad c_{p,w}^{-1} \int_{|\zeta| < \varepsilon} N_{\theta,p,w}(\zeta) dA(\zeta) \leq \|\theta'\|_{p,w}^p \leq \varepsilon^{-2} c_{p,w} \int_{|\zeta| < \varepsilon} N_{\theta,p,w}(\zeta) dA(\zeta),$$

if $1/2 < p \leq 1$, and $a_w < 2p - 2$, $b_w > -1$, or $p > 1$ and $a_w < p - 1$, $b_w > p - 2$. Moreover, in the limit case, when $1/2 < p \leq 1$, $a_w = 2p - 2$, $b_w > -1$, or

$p > 1$, $a_w = p - 1$, $b_w > p - 2$, if w satisfies the condition (*) in Proposition 1 then

$$(11) \quad \|\theta'\|_{p,w}^p \leq \varepsilon^{-2} c_{p,w} \int_{|\zeta| < \varepsilon} N_{\theta,p,w}^\ell(\zeta) dA(\zeta),$$

and

$$(12) \quad \int_{|\zeta| < \varepsilon} N_{\theta,p,w}(\zeta) dA(\zeta) \leq c_{p,w} \int_{\mathbb{D}} |\theta'(\zeta)|^p w(|\zeta|) \log \frac{1}{1-|\zeta|} dA(\zeta).$$

Proof. By Frostman's theorem [12, Theorem 6.4], for almost every $\xi \in \mathbb{D}$, the function $\theta_\xi = (\theta - \xi)(1 - \bar{\xi}\theta)^{-1}$ is a Blaschke product. If $|\xi| < \varepsilon < 1/2$ we can apply Theorem 2 (i) to obtain

$$\|\theta'\|_{p,w}^p \leq \frac{C}{\varepsilon^2} \int_{|\xi| < \varepsilon} \|\theta'_\xi\|_{p,w}^p dA(\xi) \leq \frac{c_{p,w}}{\varepsilon^2} \int_{|\xi| < \varepsilon} \left(\sum_{\theta_\xi(z)=0} (1-|z|)^{2-p} w(|z|) \right) dA(\xi),$$

and the upper estimate in (10) follows. The proof of (11) follows exactly the same steps, using Theorem 2 (iii).

To get the lower estimate in (10), assume first that $p \leq 2$, and use the standard change of variable formula [30, p. 186] together with the Schwarz-Pick lemma to obtain

$$\begin{aligned} \int_{|\zeta| < \varepsilon} N_{\theta,p,w}(\zeta) dA(\zeta) &= \int_{|\theta| < \varepsilon} |\theta'(\zeta)|^2 w(|\zeta|) (1-|\zeta|)^{2-p} dA(\zeta) \\ &\leq \int_{\mathbb{D}} |\theta'(\zeta)|^p w(|\zeta|) dA(\zeta). \end{aligned}$$

If $p > 2$ we have as above

$$\begin{aligned} &C\varepsilon^2 \int_{|\zeta| < \varepsilon} N_{\theta,p,w}(\zeta) dA(\zeta) \\ &\leq \int_{|\xi| < \varepsilon} \int_{|\zeta| < \varepsilon} \left(\frac{(1-|\xi|^2)(1-|\zeta|^2)}{|1-\bar{\xi}\zeta|^2} \right)^{p-2} N_{\theta,p,w}(\zeta) dA(\zeta) dA(\xi) \\ &= \int_{|\xi| < \varepsilon} \int_{|\theta| < \varepsilon} (1-|\theta_\xi(\zeta)|^2)^{p-2} |\theta'(\zeta)|^2 w(|\zeta|) (1-|\zeta|)^{2-p} dA(\zeta) dA(\xi). \end{aligned}$$

By Hölder's inequality and Proposition 1 (ii) we obtain

$$\begin{aligned}
& C\varepsilon^2 \int_{|\zeta| < \varepsilon} N_{\theta,p,w}(\zeta) dA(\zeta) \\
& \leq \pi \varepsilon^p \|\theta'\|_{p,w}^2 \left(\int_{|\xi| < \varepsilon} \int_{\mathbb{D}} \frac{(1 - |\theta_\xi(\zeta)|^2)^p}{(1 - |\zeta|)^p} w(|\zeta|) dA(\zeta) dA(\xi) \right)^{\frac{p-2}{p}} \\
& \leq \pi \varepsilon^p \|\theta'\|_{p,w}^2 \left(-2 \int_{|\xi| < \varepsilon} \int_{\mathbb{D}} \log |\theta_\xi| (1 - |\zeta|)^{-p} w(|\zeta|) dA(\zeta) dA(\xi) \right)^{\frac{p-2}{p}} \\
& \leq C' \varepsilon^p \|\theta'\|_{p,w}^2 \left(\int_{|\xi| < \varepsilon} N_{\theta,p,w}(\xi) dA(\xi) \right)^{\frac{p-2}{p}}.
\end{aligned}$$

Finally, (12) follows by the same argument. \square

There are two remarks to be made here. The first one is the following interesting corollary.

COROLLARY 2. *Let $(p_1, p_2) \in (1/2, 1] \times (1/2, 1] \cup (1, \infty) \times (1, \infty)$, with $0 < p_2 - p_1 < 1$, and assume that the weight w satisfies $b_w > -1$, $a_w < 2p_1 - 2$, when $1/2 < p_1 < p_2 \leq 1$, and $p_2 - 2 < b_w \leq a_w < p_1 - 1$ when $p_1, p_2 > 1$. Then there exists a constant $c > 0$ depending only on p_1, p_2 and w such that for every inner function θ we have*

$$c^{-1} \|\theta'\|_{p_2,w}^{p_2} \leq \int_{\mathbb{D}} |\theta'(z)|^{p_1} w(z) (1 - |z|)^{p_1 - p_2} dA(z) \leq c \|\theta'\|_{p_2,w}^{p_2}.$$

Proof. If $w_1(z) = w(z)(1 - |z|)^{p_1 - p_2}$ then p_1, w_1 satisfy the conditions in Theorem 3 and

$$N_{\theta,p_1,w_1}(\zeta) = N_{\theta,p_2,w}(\zeta).$$

\square

Note that the inequality on the left follows by the Schwarz-Pick lemma, but the one on the right is somewhat surprising.

Secondly, we should say a few words about the conditions on w and p . The point we want to make is that they place us in an interesting type of spaces. For example, if w is a standard weight, $w(z) = (\alpha + 1)(1 - |z|)^\alpha$, $\alpha > -1$, then $a_w = b_w = \alpha$, and the assumptions are

$$\alpha < 2p - 2, \quad \text{if } 1/2 < p \leq 1, \quad p - 2 < \alpha < p - 1, \quad \text{if } p > 1.$$

If $\alpha > p - 1$, $p > 1/2$ then the derivative of an arbitrary H^2 -function belongs to $L_a^{p,\alpha} = L_a^p(w)$, and if $2 \geq p > 1$, $\alpha < p - 2$, only the derivative of a finite Blaschke product can belong to this space.

3. THE USE OF DUALITY

If $p > 1$, the conditions $p - 2 < b_w \leq a_w < p - 1$, considered in the previous section, ensure that the dual of $L_a^p(w)$ can be identified with $L_a^q(w^{-q/p})$, $\frac{1}{p} + \frac{1}{q} = 1$ (see [22]), with respect to the unweighted Bergman pairing

$$(f, g) = \int_{\mathbb{D}} f \bar{g} dA,$$

i.e., the $L_a^p(w)$ -norm can be estimated as

$$(13) \quad \|f\|_{p,w} \asymp \sup_{\|g\|_{q,w^{-q/p}} \leq 1} \left| \int_{\mathbb{D}} \bar{f} g dA \right|.$$

This information proves useful, for it provides an additional tool for estimating the $L_a^p(w)$ -norm of derivatives of Blaschke products. By a direct computation based on the Cauchy-Green (Stokes) formula [10, p. 17], we have

$$(14) \quad \int_{\mathbb{D}} \bar{f}' g dA = \frac{1}{2i} \int_{|z|=1} \bar{f}(z) g(z) dz, \quad f, g \in H^2.$$

An immediate application of (14) is the following result related to the so-called \mathcal{F} -property.

PROPOSITION 2. *If $p > 1$, and $p - 2 < b_w \leq a_w < p - 1$, there exists a positive constant $c_{w,p}$ depending only on p and w such that whenever θ_1, θ_2 are inner functions with $\theta_2/\theta_1 \in H^\infty$, we have*

$$\|\theta_1\|_{p,w} \leq c_{p,w} \|\theta_2\|_{p,w}.$$

Proof. By (14) we have that

$$\int_{\mathbb{D}} \bar{\theta}_1' g dA = \int_{\mathbb{D}} \bar{\theta}_2' \frac{\theta_1}{\theta_2} g dA,$$

whenever $g \in H^2$. Since H^2 is dense in $L_a^p(w)$, the result follows by (13). \square

If $f = B$ is a finite Blaschke product with simple zeros, then using the fact that $\bar{B}(z) = 1/B(z)$ when $|z| = 1$, we can evaluate the line integral in (14) with help of the residue formula. Since H^2 is dense in $L_a^q(w^{-q/p})$ we have in this case

$$(15) \quad \frac{1}{2i} \int_{|z|=1} \bar{B}(z) g(z) dz = \pi \sum_n \frac{1 - |z_n|^2}{B_n(z_n)} g(z_n), \quad g \in L_a^q(w^{-q/p}),$$

where $(z_n)_{n=1}^\infty$ are the zeros of B and, as before, B_n denotes the Blaschke product with zeros $\{z_k : k \neq n\}$. We will also use the notation $B^{[N]}$ for the N -th partial product of the Blaschke product B .

PROPOSITION 3. Assume that $p > 1$, and $p - 2 < b_w \leq a_w < p - 1$. Then:

(i) For every Blaschke product B with simple zeros we have

$$\|B'\|_{p,w} \asymp \limsup_{N \rightarrow \infty} \sup_{\|g\|_{q,w^{-q/p}} \leq 1} \left| \sum_{n=1}^N \frac{1 - |z_n|^2}{B_n^{[N]}(z_n)} g(z_n) \right|,$$

where the constants involved depend only on p and w .

(ii) If the zero sequence $(z_n)_{n=1}^\infty$ is interpolating for $L_a^q(w^{-q/p})$ then

$$\|B'\|_{p,w}^p \asymp \sum_n \frac{(1 - |z_n|)^{2-p} w(|z_n|)}{|B_n(z_n)|^p}.$$

In particular, this holds whenever B is an interpolating Blaschke product.

Proof. (i) By Proposition 2 it follows immediately that

$$\limsup_{N \rightarrow \infty} \sup_{\|g\|_{q,w^{-q/p}} \leq 1} \left| \sum_{n=1}^N \frac{1 - |z_n|^2}{B_n^{[N]}(z_n)} g(z_n) \right| \leq C \|B'\|_{p,w}.$$

Conversely, let g be a polynomial such that $\|g\|_{q,w^{-q/p}} \leq 1$ and

$$\|B'\|_{p,w} < C' \left| \int_{\mathbb{D}} \overline{B'} g dA \right|,$$

where C' depends only on p and w . Since

$$\int_{\mathbb{D}} \overline{B'} g dA = \lim_{N \rightarrow \infty} \int_{\mathbb{D}} \overline{(B^{[N]})'} g dA,$$

the result follows.

(ii) If $(z_n)_{n=1}^\infty$ is interpolating for $L_a^q(w^{-q/p})$ then

$$\begin{aligned} \sup_{\|g\|_{q,w^{-q/p}} \leq 1} \left| \sum_{n=1}^N \frac{1 - |z_n|^2}{B_n^{[N]}(z_n)} g(z_n) \right| &\asymp \sup_{\|(x_n)\|_{\ell^q} \leq 1} \left| \sum_{n=1}^N \frac{(1 - |z_n|^2)^{1-2/q} w^{1/p}(|z_n|)}{B_n^{[N]}(z_n)} x_n \right| \\ &= \sum_{n=1}^N \frac{(1 - |z_n|)^{2-p} w(|z_n|)}{|B_n^{[N]}(z_n)|^p}, \end{aligned}$$

and the result follows by the monotone convergence theorem. \square

It is interesting to note that part (ii) together with Theorem 2 (i) yield an average lower bound for $|B_n(z_n)|$ in the case when $(z_n)_{n=1}^\infty$ is interpolating for $L_a^q(w^{-q/p})$. We have that

$$(16) \quad \sum_n \frac{(1 - |z_n|)^{2-p} w(|z_n|)}{|B_n(z_n)|^p} \leq C \sum_n (1 - |z_n|)^{2-p} w(|z_n|) |B_n(z_n)|^p.$$

We now turn to Cohn's theorem [7] which asserts that for interpolating Blaschke products B and $s \in (1/2, 2/3)$ we have

$$K'_B \subset H^s \iff B' \in H^{\frac{2s}{2-s}},$$

where $K_B = H^2 \ominus BH^2$, and K'_B denotes the set of derivatives of functions in K_B . Given a Blaschke product B and $p > 1$ we denote by

$$K_B^p = H^p \cap \overline{BH_0^p} = (BH^q)^\perp, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and by $(K_B^p)'$ the set of derivatives of functions in K_B^p .

THEOREM 4. *Let B be an interpolating Blaschke product, let $p > 1$, and let w be a normal weight. Let $1 < s < p$ and assume that $s - 2 + \frac{s}{p} < b_w \leq a_w < s - 1$.*

Then $(K_B^p)' \subset L_a^s(w)$ if and only if $B' \in L_a^{s\frac{p-1}{p-s}}(w^{\frac{p}{p-s}})$.

Proof. Since $(z_n)_{n=1}^\infty$ is an interpolating sequence, a function $f \in H^p$ belongs to K_B^p if and only if it can be written as

$$f(z) = \sum_n c_n \frac{(1 - |z_n|)^{1/q}}{1 - \bar{z}_n z},$$

where the numbers $c_n \in \mathbb{C}$ are uniquely determined by f and z_n , $n = 1, 2, \dots$, are the zeros of B . Moreover,

$$\|f\|_p^p \asymp \sum_n |c_n|^p.$$

Now let $r > 1$ be given by $\frac{1}{r} = \frac{1}{s} - \frac{1}{p}$ and denote by r' and s' the conjugate indices of r and p , respectively; that is,

$$\frac{1}{r'} + \frac{1}{r} = \frac{1}{s'} + \frac{1}{s} = 1.$$

By the condition on w , we see that we can apply (13) to obtain

$$\begin{aligned} & \sup\{\|f'\|_{s,w} : f \in K_b^p, \|f\|_{H^p} \leq 1\} \\ & \asymp \sup\left\{\left|\int_{\mathbb{D}} f' \bar{g} dA\right| : \|g\|_{s',w^{-s'/s}} \leq 1, f \in K_b^p, \|f\|_{H^p} \leq 1\right\} \\ & \asymp \sup\left\{\left|\sum_n \bar{z}_n c_n \overline{g(z_n)} (1 - |z_n|)^{1/q}\right| : \|g\|_{s',w^{-s'/s}} \leq 1, \sum_n |c_n|^p \leq 1\right\}. \end{aligned}$$

Now recall that, by Theorem 1, $(z_n)_{n=1}^\infty$ is interpolating for $L_a^{s'}(w^{-s'/s})$, hence

$$\begin{aligned} & \sup\left\{\left|\sum_n \bar{z}_n c_n \overline{g(z_n)} (1 - |z_n|)^{1/q}\right| : \|g\|_{s',w^{-s'/s}} \leq 1, \sum_n |c_n|^p \leq 1\right\} \\ & \asymp \sup\left\{\left|\sum_n a_n c_n w(|z_n|)^{1/s} (1 - |z_n|)^{1/q-2r/s'}\right| : \sum_n |a_n|^{s'} \leq 1, \sum_n |c_n|^p \leq 1\right\}. \end{aligned}$$

With our notations, we can rewrite this as

$$\sup\{\|f'\|_{s,w}^r : f \in K_b^p, \|f\|_{H^p} \leq 1\} \asymp \sum_n w(|z_n|)^{r/s} (1 - |z_n|)^{r/q-2r/s'},$$

and since $\frac{1}{q} - \frac{1}{s'} = \frac{1}{r}$, we obtain

$$\sup\{\|f'\|_{s,w}^r : f \in K_b^p, \|f\|_{H^p} \leq 1\} \asymp \sum_n w(|z_n|)^{r/s} (1 - |z_n|)^{2-r/q}.$$

Finally, from the fact that

$$a_{w^{r/s}} = \frac{r}{s} a_w, \quad b_{w^{r/s}} = \frac{r}{s} b_w,$$

we see that the conditions in the statement are equivalent to

$$\frac{r}{q} - 2 < b_{w^{r/s}} \leq a_{w^{r/s}} < \frac{r}{q} - 1,$$

and since $\frac{r}{q} > 1$, Theorem 2 yields

$$\sum_n w(|z_n|)^{r/s} (1 - |z_n|)^{2-r/q} \asymp \int_{\mathbb{D}} |B'|^{r/q} w^{r/s} dA,$$

which completes the proof. \square

We should point out here that the case when $s \geq p$ can be treated similarly, but it leads either to the limit cases in Theorem 2, or to trivial results.

4. DERIVATIVES OF HIGHER ORDER

It is well known that

$$\int_{\mathbb{D}} |f(z)|^p dA(z) \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{np} dA(z)$$

and it turns out that this extends to weighted Bergman spaces with normal functions as well.

THEOREM A. *If $b_w < 1$ then*

$$\|f\|_{p,w}^p \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{np} w(|z|) dA(z).$$

In the unweighted case $w = 1$, the above statement is a well known theorem of Hardy and Littlewood (cf. [9, Chapter 5] where one inequality is proved, and the other can easily be proved using the techniques displayed there). The above generalization was proved explicitly in [5] for $p \geq 1$ and in [19] for $p < 1$.

Taking into account that the radial function w is normal if and only if the associated function $W(z) = (1 - |z|)^p w(z)$ is normal, one can derive from Theorem A various analogues of our main theorems for derivatives of higher order. Just to illustrate a flavor of such results, we now state without proof one such statement.

THEOREM 5. *Let w be a normal function and let $1/2 < p < \infty$.*

- (i) Assume that $a_w < 2p - 2$, $b_w > -1$, if $1/2 < p \leq 1$, and that $a_w < p - 1$, $b_w > p - 2$, if $p > 1$. Then there exists a positive constant $c_{p,w}$ such that for every Blaschke product B we have

$$\sum_{k=0}^{n-1} |B^{(k)}(0)|^p + \|B^{(n)}\|_{p,w}^p \leq c_{p,w} \sum_{B(z)=0} (1 - |z|)^{2-np} w(|z|).$$

- (ii) Suppose that the zero set of the Blaschke product B is separated with separation constant $\delta > 0$. Suppose that either $a_w < 2p - 2$, $b_w > -1$, when $1/2 < p \leq 1$, or $a_w < p - 2$, $b_w > -1$, when $p > 1$. Then, in both cases, there exists a positive constant $c_{p,w,\delta}$ such that

$$\sum_{B(z)=0}^{\infty} (1 - |z|)^{2-np} w(|z|) \leq \sum_{k=0}^{n-1} |B^{(k)}(0)|^p + c_{p,w,\delta} \|B^{(n)}\|_{p,w}^p.$$

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